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CENTERS OF GENERIC DIVISION ALGEBRAS, THE RATIONALITY PROBLEM 1965-1990

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ABSTRACT

A survey is given of the rationality problem of the center of generic division algebras. Connections are given with Brauer groups of fields, geometric moduli problems and representation theory. An outline is given of recent results.

1. Rough Guide

Finding a suitable starting point for a survey is an art in itself, usually heavily determined by the place and time of the conference. In this particular case, the best start I could come up with was Chicago in 1965. The occasion being that S.A. Amitsur gave a series of lectures at the University of Chicago while C. Procesi was a student of I.N. Herstein preparing there for his PhD. The collision between these two great algebraists resulted in the rings we now know as the ring of generic n by n matrices, $R_n = k < X, Y >_n$ and the corresponding generic division algebra $D_n = k(X, Y)_n$.

Ten years before, Amitsur [1] proved that the quotient of the free algebra in (say) two variables modulo the T-ideal of all identities satisfied by couples of $n \times n$ matrices is an Ore domain and hence has a classical division ring of quotients. C. Procesi was able to give a more down to earth interpretation of this ring as the ring of 2 generic n by n matrices:

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Consider the commutative polynomial ring $P_n = k[x_{ij}, y_{ij} : 1 \le i,j \le n]$ and define the generic matrices

$$X = (x_{ij})_{i,j} \text{ and } Y = (y_{ij})_{i,j} \in M_n(P_n).$$

Then, Amitsur's ring coincides with the k-subalgebra R_n of $M_n(P_n)$ generated by X and Y and is called the ring of (two) generic n by n matrices. These rings as well as their division rings of quotients (the generic division algebras D_n) and their centers $C_n = Z(D_n)$ were studied by Procesi in his Ph.D. (University of Chicago, 1966), part of which was published in [22], one of three papers indispensible to read when one is interested in p.i.-theory. This paper contains the first explicit mention of the rationality problem but accompanied with heavy scepticism :

"We have met difficulties in finding a precise internal description of C_n in general, and so, in particular, we leave as open the question of deciding whether C_n is rational or not over k ... the evidence is very slim, except of course for 2×2 matrices where everything expected can be proved" [22, pp. 240–241].

In this note we will describe the little progress made on this problem over the last 25 years. Before we give a preview of things to come, let us formalize some definitions :

Definition 1: Let $k \subset L$ be commutative fields, then L is said to be
1. **rational** iff $L \simeq k(x_1, ..., x_l)$ the purely transcendental field in l variables for some l, the isomorphism being one of k-algebras
2. **stably rational** iff $L(y_1, ..., y_r)$ for some r becomes rational over k i.e. is isomorphic as k-algebras to $k(x_1, ..., x_s)$ for some s
3. **retract rational** iff there is a k-affine domain R having L as its functionfield, an embedding $i : R \hookrightarrow k[x_1, ..., x_s]_f$ into a localization of a polynomial ring over k and a retract $\pi : k[x_1, ..., x_s]_f \to R$ s.t. $\pi \circ i = 1_R$
4. **unirational** iff L is a subfield in a rational field. Note that we can change the embedding and the field such that L becomes of finite index. ∎

Two k-fields K and L are called **stably equivalent** iff $K(x_1, ..., x_r) \simeq L(y_1, ..., y_s)$ as k-algebras for some r and s

Clearly, we have the implications :

$$\text{rational} \Rightarrow \text{stably rational} \Rightarrow \text{retract rational} \Rightarrow \text{unirational}$$

and although some of the inverse implications were major open problems at a time (e.g. Lüroth's problem asked whether unirationality implies rationality and Zariski's problem asked whether stable rationality implies rationality) all inverse implications are now known to be false, in general.

In Table 1, we summarize the present knowledge on the different rationality criteria for C_n:

TABLE 1. Results on C_n up to 1990

unirational	all n	Procesi	1967	[22]
retract rational	n squarefree	Saltman	1984	[28]
stably rational	$n \mid 420$	Katzylo	1989	[15]
		Schofield	1989	[32]
		Bessenrodt, Le Bruyn	1989	[6]
rational	$n = 2$	Sylvester	1883	[34]
		Procesi	1967	[22]
	$n = 3$	Formanek	1979	[13]
	$n = 4$	Formanek	1980	[14]

The motivation for studying rationality of C_n changed drastically over the years.

Around 1970 the main motivation was to find a "natural" counterexample to the Lüroth problem, for Procesi [22] did prove that C_n is always unirational whereas it was (and still is) not clear at all whether it is rational. However, a few years later one found lower dimensional counterexamples by an entirely different method, e.g. [5].

Around 1980 renewed interest in the rationality problem was fueled by the implications (stable) rationality has on problems concerning the Brauer group of a field, in particular whether it is generated by cyclic algebras (provided the field has enough roots of unity). But again, a few years later Merkurjev and Suslin [20] were able to settle this problem affirmatively by an entirely different method.

Today, the main motivation for studying rationality of C_n comes from its connection with geometrical moduli problems. It did turn out that many wildly open rationality problems can be reduced up to stable equivalence to that of the center of the generic division algebra.

A more serious obstacle an unprepared reader may run into is that there at least three totally different approaches to the study of C_n, pursued by people with very different research interests and even aiming at different answers.

As we go along we will explain in more detail all terminology contained in Table 2.

TABLE 2. Rough map to the literature

Description	$Z(D_n)$	$F(M_n \oplus M_n)^{\mathrm{PGL}_n}$	$F(U_n \oplus G_n)^{S_n}$
theme	Brauer groups	Geometry	Representation theory
answer	Negative	Reductions	Positive
trick	Saltman lift	Bogomolov transfer	Formanek strategy
best result	retract rat. n squarefree	Schofield-Katzylo reduction	rational $n < 4$ stably rat $n = 5, 7$

After this birds eye view on the territory ahead, we will now consider each of the three main themes in a more systematic manner :

2. The Brauer Group Theme

The key idea here is easy to grasp: as C_n is the center of the generic division algebra D_n, knowledge on C_n and on D_n should imply knowledge on central simple algebras of dimension n^2 over their centers $L \supset k$ and hence on the Brauer group $\mathrm{Br}(L)$ of any field extension L of k.

The first result in this direction was obtained in 1972 by S.A. Amitsur [2] in giving a counterexample to the famous crossed product problem .If Δ is a division algebra of dimension n^2 over its center K, then it is well known that every maximal commuative subfield of Δ has dimension n over K. Moreover, one can prove that among all those different fields there always are separable extensions of K. The crossed product problem asks whether there is always also a Galois extension among them. In general, the solution was (and is) only known to be positive for $n \mid 12$.

Amitsur proved that D_n cannot have a Galois maximal subfield for $k = \mathbb{Q}$ if $p^2 \mid n$ for p an odd prime or if $8 \mid n$. His proof has two parts : first he uses genericity of D_n to show that if D_n contains such a Galois extension with group G, then every division algebra of dimension n^2 over any extension field of k contains a Galois extension with group G. Next, he constructs explicit examples of division algebras over extensions of $\mathbb{Q}$ with the property that all Galois maximal subfields have a metacyclic group. For $p^2 \mid n$ or $8 \mid n$ he is then able to construct examples with different groups, concluding the argument.

After this problem was settled, attention turned to another open problem namely whether the Brauer group of K is generated by cyclic algebras? Again, the answer was only known under heavy restrictions on K (e.g. if K is a numberfield by the Hasse–Brauer–Noether theorem). In 1974, S. Bloch [7] (the proof appeared only in [8]) was able to prove a K-theoretic statement of which a weak Brauer group form can be phrased as: if K contains all roots of unity and if $\mathrm{Br}(K)$ is generated by cyclic algebras then the same is true for the Brauer group of $K(x_1, ..., x_n)$ a purely transcendental field extension of K.

The results of Amitsur and Bloch were then "combined" by S. Rosset in probably, one of the more interesting erroneous papers [25]. Nevertheless, Rosset's idea has some merrit as R. Snider [33] and C. Procesi [24] showed. Whereas Snider considered another class of generic algebras, his "generic crossed products" (discovered independently around the same time by S. Rosset [26]), C. Procesi applied the idea to the generic division algebras : assume that C_n is stably rational over a field k s.t. $\mathrm{Br}(k)$ is generated by cyclics (e.g. k algebraically closed), then by Blochs result $[D_n]$ in $Br(C_n)$ is similar to a product of cyclic C_n-algebras. But then, using the generic property of D_n one can show that the same holds for any division algebra of dimension n^2 over its center $K \supset k$. The reader may see [24] for the details. Therefore, if C_n is stably rational for all n then the Brauer group of any over-field of k would have to be generated by cyclics. As most specialists were rather sceptical about the generated-by-cyclics-conjecture, finding a specific field s.t. a division algebra of dimension n^2 was not similar to a product of cyclics would prove that C_n could not be (stably) rational.

It came therefore as a surprise when Merkurjev and Suslin were able to show that the conjecture is true whenever the field contains enough roots of unity [20].

This result seemed to finish this approach to the rationality problem until D. Saltman entered the picture [28]. He showed that rationality (even retract rationality) has much stronger consequences than the Brauer group being generated by cyclic algebras.

His key observation was that retract rationality for certain generic objects in fairly general classes of algebras is equivalent to a lifting property for these algebras. If we specialize to the situation at hand, i.e. the generic division algebra D_n and the class of all central simple algebras of dimension n^2 over their centers, then his result can be phrased as follows :

C_n is retract rational if and only if for every local domain A with maximal ideal m and any c.s.a. Δ of dimension n^2 over the residue field A/m there exists an Azumaya algebra Λ with center A s.t. $\Delta \simeq \Lambda \otimes_A (A/m)$.

Hence,the Saltman-lift idea to disprove rationality (or even retract rationality) of C_n is to construct manageable local domains failing to possess this lifting property.

On the positive side, assume that for a certain number n one can prove the lifting property, then one has proved retract rationality of C_n as Saltman was able to do for n squarefree (see [28] for the prime number case).

3. The Geometrical Theme

Here, the key idea is to view C_n as the functionfield of a quotient variety under a reductive group action and to hope that the powerful machinery of geometric invariant theory will allow us to reduce the problem to more manageable ones.

The starting point this time is a paper by M. Artin [4], the second of three papers indispensible to read if you are (still) interested in p.i.-theory. Artin not only characterizes Azumaya algebras by their identities in this paper, but he also proves the following description of C_n:

Consider pairs of $n \times n$ matrices over an algebraically closed field k of characteristic zero

$$M_n(k) \oplus M_n(k) = V_n.$$

The general linear group $\mathrm{GL}_n(k)$ acts on this vectorspace by simultaneous conjugation, thereby making it really an action of the projective linear group $\mathrm{PGL}_n(k)$ $= \mathrm{GL}_n(k)/k^*$. Hence, $\mathrm{PGL}_n(k)$ acts on the coordinate ring of V_n which is the polynomial ring P_n introduced before and one can consider the invariant algebra under this action:

$$P_n^{\mathrm{PGL}_n} = \{f \in P_n : \gamma \cdot f = f, \forall \gamma \in (\mathrm{P})\mathrm{GL}_n\}.$$

By general theory, this algebra is an affine k-algebra and it is the coordinate ring of the quotient variety V_n/PGL_n parametrizing the closed orbits of the action (which in this case are precisely the isoclasses of n-dimensional semi-simple representations of the free algebra $k < X, Y >$ [4]).

Although Artin was not able to prove that $R_n^{\mathrm{PGL}_n}$ is generated by traces $\mathrm{Tr}(X^{i_1}Y^{j_1}....X^{i_k}Y^{j_k})$ for all k, i_l and j_l (a fact which was later proved by Procesi [23]) he did prove that the field of fractions of $R_n^{\mathrm{PGL}_n}$ (and hence the functionfield of V_n/PGL_n) is equal to C_n.

This paper initiated the extensive study of finite dimensional representations of affine p.i.-rings but had no immediate bearing on the rationality problem itself until quite recently M. Van den Bergh [35] and myself [18] used Artin's description of C_n to link it to geometrical questions. Van den Bergh proved that C_n is the functionfield of the generic Jacobian variety of plane curves of degree n (and using this interpretation he gave a nice geometrical proof of the rationality of C_3). Later (although the paper appeared first) I proved that C_n is the functionfield of the moduli space $M(n; 0, n)$ of stable rank n vector bundles on the projective plane with Chern numbers $c_1 = 0, c_2 = n$. This last fact was recently extended by P.I. Katzylo [16] to that of stable rank r bundles on $\mathbb{P}_2$ with Chern numbers $c_1 = 0, c_2 = s$.He proved that the functionfield of the corresponding moduli space is stably equivalent to C_n where $n = \gcd(r, s)$.

Having reduced our problem to an entirely geometrical one (rationality problems for moduli spaces have been studied thoroughly), one could try to skim the geometry literature for clues. And in fact, Maruyama [19] claimed to have proved stable rationality of $M(n; 0, n)$ by showing that it is stably rational over $k(M_n(k))^{\mathrm{PGL}_n} = k(\mathrm{Tr}(X), \mathrm{Tr}(X^2), ..., \mathrm{Tr}(X^n))$. However, if geometers would have taken the trouble to read the related ring theory literature they would have known by 1980 that such a result could not be true for $n = 4$ as an argument due to R. Snider mentioned in [14] and [24] shows. The gap in the Maruyama proof was communicated by K. Hulek, J. Le Potier and independently by D. Saltman, see e.g. [18].

The next step forward was a general transfer result due to Bogomolov, see e.g. [12]. Let G be any reductive group and V and W two finite dimensional G-representations which are almost free, meaning that the stabilizer of a sufficiently general point is trivial. Then, if k is algebraically closed of characteristic zero, Bogomolov shows that the respective invariant fields $k(V)^G$ and $k(W)^G$ are stably equivalent.

This Bogomolov transfer result shows us two things : (a) stable rationality of invariant fields of almost free representations is a property of the group rather than the particular representation and (b) as long as we are interested in stable rationality we have some freedom in chosing another (almost free) representation.

Using this result and the theory of reflexion functors in the representation theory of quivers, due to Berstein, Gelfand and Ponomarov, A. Schofield and myself [17] were able to reduce the stable rationality problem for many moduli problems which can be expressed in linear data to that of C_n. Rather than stating the result in its precise form (which involves some quiver-lingo) let me give a characteristic example, the r-subspace problem:

This problem asks for the determination of all possible positions of r subspaces of given dimensions say $n_1, ..., n_r$ in a big vectorspace of dimension N up to basechange in this large space. The corresponding geometrical problem is that of classifying GL_N-orbits in a product of Grassmann varieties

$$\mathrm{Grass}(n_1, N) \times \cdots \times \mathrm{Grass}(n_r, N)$$

and it was one of the test-examples for Mumford's geometric invariant theory [21]. He proved that a point $(U_1, ..., U_r)$ in this product is a stable point w.r.t. the GL_N-action iff one cannot find a proper subspace V of dimension v of k^N s.t.

$$v^{-1}(\sum_{i=1}^{r} \dim_k(V \cap U_i) \geq N^{-1} \sum_{i=1}^{r} \dim_k(U_i) = N^{-1} \sum_{i=1}^{r} n_i.$$

Now, assume that one does have a stable point, then our result implies that the functionfield of the corresponding quotient variety is stably equivalent to C_n where $n = \gcd(N, n_1, ..., n_r)$.

Another, more recent, application of Bogomolov transfer is the Schofield–Katzylo reduction result [32], [15]. They prove that if $n = a.b$ with $(a, b) = 1$ then C_n is stably equivalent to $C_a \otimes C_b$ reducing the (stable) rationality problem of C_n to that of prime power values of n.

Schofield's proof contains two parts: first it is well known that principal PGL_n-bundles (in the étale topology) correspond to Azumaya algebras of rank n^2 and hence give rise to central simple algebras of dimension n^2 over their centers. If $n = a.b$ with $(a, b) = 1$ such a central simple algebra can be written in a controllable way as a tensor product of central simple algebras of dimension a^2

and b^2. Therefore, one expects a reduction of group argument from PGL_n to $\mathrm{PGL}_a \times \mathrm{PGL}_b$ and Schofield shows that the generic fibre of the natural map of homogenous spaces

$$\mathrm{GL}_{n^2}/\mathrm{PGL}_a \times \mathrm{PGL}_b \to \mathrm{GL}_{n^2}/\mathrm{PGL}_n$$

is indeed stably rational. The next step is then to use Bogomolov transfer to show that $\mathrm{GL}_{n^2}/\mathrm{PGL}_a \times \mathrm{PGL}_b$ is stably equivalent to $V_a/\mathrm{PGL}_a \times V_b/\mathrm{PGL}_b$ finishing the proof.

4. The Representation Theoretical Theme

This time we view C_n as a field of tori-invariants under a finite group and use the extensive theory of tori-invariants developed in the early seventies to reduce the problem to one of lower transcendence degree and in some cases to prove (stable) rationality.

Once again, the starting point is Procesi's paper [22] in which he proved unirationality of C_n by showing that the field obtained by adjoining to C_n the characteristic roots of the first generic matrix X is rational over k and has a natural action of the symmetric group S_n such that C_n is its field of invariants.

Later, Formanek [13] gave an interpretation of this fact in terms of tori-invariants. Recall that for a finite group G a G-lattice is a free Abelian group of finite rank with a G-action and if it has a $\mathbb{Z}$-basis which is permuted by this action, it is called a permutation G-lattice. Let $U_n = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_n$ be the standard permutation S_n-lattice $\mathbb{Z}S_n/S_{n-1}$. Further, let $H_n = \mathbb{Z}b_{11} \oplus \mathbb{Z}b_{12} \oplus \cdots \oplus \mathbb{Z}b_{nn}$ be the permutation S_n-lattice $\mathbb{Z}S_n/S_{n-2} \oplus U_n$ via the action $\sigma.b_{ij} = b_{\sigma.i,\sigma.j}$. Then, there is a morphism of S_n-lattices $\phi : H_n \to U_n$ determined by $\phi(b_{ij}) = u_i - u_j$ giving rise to an exact sequence

$$0 \to G_n \to H_n \to U_n \to \mathbb{Z} \to 0$$

where $G_n = \mathrm{Ker}(\phi)$ and the rightmost map $\pi : U_n \to \mathbb{Z}$ is the augmentation map sending each u_i to 1 (note that the kernel of π is the root lattice A_{n-1}).

If G is a finite group and if L is a field with a faithful G-action by k-automorphisms and if A is a G-lattice, then there is a composite G-action on the group algebra $L[A]$ of the Abelian group A over L and hence also on its field of fractions $L(A)$. The field of invariants $L(A)^G$ is then called the field of tori-invariants of L and A under G.

In our case, there is a faithful S_n-action on $k(U_n)$ and we can form the field of tori-invariants $k(U_n)(G_n)^{S_n} = k(U_n \oplus G_n)^{S_n}$. The Procesi–Formanek result now states that this field coincides with C_n.

The usefulness of this description is clear from the following result: define two S_n lattices M and N to lie in the same **bag** iff there is a third S_n-lattice E and exact sequences of S_n-lattices

$$0 \to M \to E \to P_1 \to 0$$

$$0 \to N \to E \to P_2 \to 0$$

with the P_i permutation S_n-lattices. Now, for any faithful S_n-lattice F we have that $k(F \oplus M)^{S_n}$ is stably equivalent to $k(F \oplus N)^{S_n}$ (over $k(F)^{S_n}$) if and only if M and N belong to the same bag.

This leads us directly to Formanek's strategy to prove stable rationality of C_n implicit in [14]: find a faithful S_n-lattice B_n of low rank lying in the same bag as G_n (which itself has rank $n^2 - n + 1$) and prove that $k(B_n)^{S_n}$ is (stably) rational over k.

Indeed, the result mentioned before states that $C_n = k(U_n \oplus G_n)^{S_n}$ is stably equivalent to $k(U_n \oplus B_n)^{S_n}$. As U_n is a permutation lattice it follows from an old result of Masuda that $k(U_n \oplus B_n)^{S_n}$ is rational over $k(B_n)^{S_n}$ which by assumption is (stably) rational over k, hence so will be C_n.

This reduces our problem partly to integral representation theory of the symmetric groups. Whereas $\mathbb{Q}S_n$-modules are completely determined by their characters, $\mathbb{Z}S_n$-lattices in general do not decompose uniquely into indecomposables and there are in general many nonisomorphic S_n-lattices lying in the same rational representation. These two facts make integral representations of S_n a lot harder to study than the usual representations.

Still, by the Jordan–Zassenhaus result there are only a finite number of isoclasses of S_n-lattices for any given rank and so one could in principle hope to give a full bag-picture of the S_n-lattices up to a certain rank. In practice however, things turn out to be quite hopeless for ranks greater than n. If the rank is smaller than $n-1$, then the rational representation is made up of copies of the trivial representation corresponding to the Young tableau $(1, ..., 1)$ and of the sign representation corresponding to the tableau (n) so these lattices can be described as extensions with building blocks $\mathbb{Z}$ and $\mathrm{sgn}(\mathbb{Z})$ and they all lie in the bag containing all permutation S_n-lattices.

If the rank equals $n-1$, then two new simple rational representations occur, namely $(2,1,1,...,1)$ and $(n-1,1)$, and one has a complete classification of all S_n-lattices contained in them, see [11]:

$$\text{In } (2,1,...,1): A_{n-1} \subset A_{n-1}[r] \subset A_{n-1}^*,$$

$$\text{In } (n-1,1): SA_{n-1} \subset SA_{n-1}[r] \subset SA_{n-1}^*,$$

with one isoclass for each divisor r of n. Here $(-)^*$ denotes the dual lattice, SA_{n-1} is the signed root lattice, i.e. the kernel of (signed) augmentation map on the signed permutation lattice SU_n which has a basis u_i on which S_n acts via $\sigma.u_i = \operatorname{sgn}(\sigma)u_{\sigma(i)}$.

The easiest case would be that G_n was lying in the bag containing all permutation lattices for then we could take $B_n = U_n$ and clearly $k(U_n)^{S_n}$ is rational on the elementary symmetric functions.

Now, for $n \leq 3$ one can show that there is just one bag entailing stable rationality for C_2 and C_3 and being a bit more prudent one can even show rationality. For example $C_2 = k(\operatorname{Tr}(X), \operatorname{Tr}(Y), D(X), D(Y), \operatorname{Tr}(XY))$. For an explicit transcendence basis of C_3 see [13].

However, if $n \geq 4$ there are always infinitely many bags in the picture so we cannot use this easy way out. Note that from the above discussion it follows that a necessary and sufficient condition for G_n to lie in the permutation bag is that C_n is stably rational over $k(M_n(k))^{\mathrm{PGL}_n} = k(U_n)^{S_n}$. At first sight this is probable as one would expect to be able to take $\operatorname{Tr}(X), ..., \operatorname{Tr}(X^n)$ as part of a transcendence basis for C_n.

Surprisingly enough, R. Snider showed around 1980 that G_4 could not lie in the permutation bag. This fact was mentioned without a hint of the proof in [13] and [24] and this caused some confusion as it conflicted Maruyama's result mentioned before. The first published proof of Sniders remark is due to Colliot-Thélène and Sansuc [9] and at about the same time Saltman (in an unpublished letter [31]) extended Sniders remark to all non-squarefree values of n.

Nevertheless, Formanek [14] was able to show that G_4 lies in the same bag as the rank three S_4-lattice SA_3^* by explicitly writing down some sequences of S_4-lattices. Moreover, he did show that $k(SA_3^*)^{S_4}$ is a rational field. In this proof he made heavy use of the fact that the general quartic equation is solvable and it started rumours (e.g. [3] and [27]) that there were reasons to believe that C_5 could not be (stably) rational.

So, G_n cannot lie in the permutation bag if n is not squarefree. What about the other extreme, i.e. what if n is a prime number? After all, Saltman proved that C_n is retract rational in these cases.

A representation theoretic proof of this result was given by Colliot-Thélène and Sansuc [9]. They proved that for n prime, G_n lies in the same bag as an invertible lattice (i.e. a direct summand of a permutation lattice), which immediatly implies (via Masudas result mentioned above) that C_n is retract rational. This result was improved by Ch. Bessenrodt and myself [6] by showing that G_n itself is invertible if n is prime.

Of course, these results make one wonder whether G_n could lie in the permutation bag for prime values of n. Unfortunately, this is never the case if n is larger than 3 (see [6]) giving a prime analogue to the Snider–Saltman result. I conjecture that G_n never lies in the permutation bag if $n \geq 4$ with the possible exception of $n = 6$.

Still, invertibility of G_n was the first step in proving stable rationality for $n = 5$ and $n = 7$ [6]. The proof consists of two steps. First we prove that G_n lies in the same bag as the dual of the root lattice A^*_{n-1} for $n = 5$ and $n = 7$. Dualizing the defining sequence of G_n we have

$$0 \to A^*_{n-1} \to B_n \to G_n \to 0.$$

Now, assume that $G_n \oplus G^*_n$ is a stable permutation lattice, i.e. that there exist permutation S_n-lattices P_i s.t.

$$G_n \oplus G^*_n \oplus P_1 \simeq P_2,$$

then one would obtain the sequence

$$0 \to A^*_{n-1} \to G_n \oplus B_n \oplus P_1 \to P_2 \to 0$$

entailing that A^*_{n-1} and G_n lie in the same bag.

For this reason, the bulk of the paper [6] consists in finding a method to determine when an invertible lattice is stable permutation, and this method was then applied to prove the above claim for $n = 5$ and $n = 7$.

The next step is a lot easier, namely showing that $k(A^*_{n-1})^{S_n}$ is rational. For, we have the sequence

$$0 \to \mathbb{Z} \to U_n \to A^*_{n-1} \to 0$$

where the leftmost map sends 1 to $\sum u_i$. On the groupalgebra level this amounts to $k[A_{n-1}^{S_n}] = k[u_1^1, u_1^{-1}, ..., u_n^1, u_n^{-1}]/(u_1u_2...u_n - 1)$. As everything is S_n-equivariant we have that

$$k(A_{n-1}^*)^{S_n} = k(\sigma_1, ..., \sigma_{n-1})$$

rational on the first $n-1$ elementary symmetric functions. This finishes an outline of the proof of stable rationality of C_5 and C_7.

However, for n a prime number larger than 7 one can show that $G_n \oplus G_n^*$ is no longer stable permutation. Hence, for such n there is no lattice of rank $n-1$ lying in the same bag as G_n and we have to gain more insight into the wild part of the S_n-lattices.

5. What's Next?

Of course one would like to extend upon the results obtained so far. However, a complete answer to the question for which n the center of the generic division algebra C_n is (stably) rational seems to be at present out of reach. Meanwhile, there are some less ambitious questions which would increase our knowledge considerably. Let me finish this note by asking three specific questions, one for each theme:

BRAUER GROUP QUESTION. The precise connection between (stable) rationality of C_n and the crossed product property of D_n should be understood. For a long time the number of n's s.t. D_n was known to be a crossed product outnumbered those for which C_n was known to be stably rational. This is now reversed, so in particular one could ask whether stable rationality of C_5 and C_7 has any bearing on the problem whether division algebras of degree 5 or 7 are cyclic.

GEOMETRICAL QUESTION. Calculate new birational invariants for V_n/PGL_n. For some time [30] (or a few years later [9] in characteristic zero) it is known that the Brauer group of a smooth proper model of C_n is trivial. Recently, Colliot-Thélène and Ojanguren [10] introduced new birational invariants, which ought to be computed for C_n.

REPRESENTATION QUESTION. What is the minimal rank of a (faithful) lattice lying in the same bag as G_n? In particular, can one improve upon the best general upper bound found by Formanek [14] (i.e. $\leq n^2 - 3n + 1$) to get it linear? Recall that for $n = 4, 5$ or 7 it is known to be $n-1$ but for larger primes the rank should be greater. Of course, a more ambitious project is to find an explicit low rank lattice lying in the same bag or at least having the same local invariants as in [6].

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